Asymptotic modelling of multi-dimensional jump boundary conditions at a fluid-porous interface

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1. Introduction

We present an original asymptotic modelling [1] for a multi-dimensional unsteady viscous fluid flow and convective transfer at a fluid-porous interface. This asymptotic model is based on the fact that the thickness d of the interfacial transition region Ω_{fp} in the one-domain representation [2, 3] is very small compared to the macroscopic length scale L. The analysis leads to an equivalent two-domain representation where transport phenomena in the transition layer of the one-domain approach are represented by algebraic jump boundary conditions at a fictive dividing interface Σ between the homogeneous fluid and porous regions. These jump conditions are thus stated in [1] up to first-order in $O(d/L)$.

The originality and great interest of this new asymptotic model lies in its general and multi-dimensional character. Indeed, it is shown that all the jump interface conditions [4, 5, 6] derived for the 1D-shear flows are recovered by taking the tangential component of the asymptotic model. In that case, the comparison between the present model and the upscaled models available in the literature gives explicit expressions of the effective jump coefficients and their associated scaling. In addition, for multi-dimensional flows as in [7, 8], the general asymptotic model yields the different components of the jump conditions including a new specific equation for the cross-flow pressure jump on Σ.

2. Asymptotic model with multi-dimensional jump interface conditions

We summarize the first-order asymptotic model derived in [1] for the multi-dimensional and unsteady viscous fluid flow with coupled heat or mass transfer inside the fluid-porous system, see Figure 1. *The main assumption is that* $\eta := d/L \ll 1$ *with* $d = O(\sqrt{|\boldsymbol{K}|})$, $\boldsymbol{K}$ being the permeability tensor in Ω_{fp}.

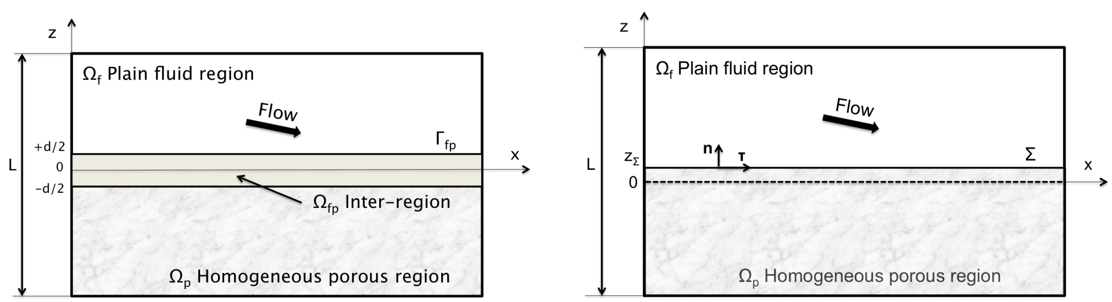

Figure 1: Fluid-porous configurations – Left: one-domain model with continuous inter-region – Right: two-domain model with a fictive interface Σ at $\xi := z_\Sigma/d$.

The jump and mean quantity of any unknown ψ on Σ are defined as in [9, 7] respectively by :

$$[\![\psi]\!]_\Sigma := \psi^+ - \psi^- = \left(\psi^f - \psi^p\right)_{|\Sigma}, \quad \text{and} \quad \overline{\psi}_\Sigma := \frac{1}{2}\left(\psi^+ + \psi^-\right) = \frac{1}{2}\left(\psi^f + \psi^p\right)_{|\Sigma}.$$

For any physical property k, the arithmetic and harmonic averages over the thickness of Ω_{fp} are :

$$< k > (x) := \frac{1}{d(x)} \int_{-d/2}^{d/2} k(x,z)\,\mathrm{d}z, \quad < k >^h (x) := \left(\frac{1}{d(x)} \int_{-d/2}^{d/2} \frac{\mathrm{d}z}{k(x,z)}\right)^{-1} = \frac{1}{< \frac{1}{k} >}.$$

The set of equations corresponds to a Stokes/Darcy-Brinkman transmission problem with source terms inside $\Omega_f \cup \Omega_p$, coupled with the advection-diffusion-reaction equation for the temperature θ :

$$\left\{\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{v} &= q && \text{in } \Omega_f \cup \Omega_p \times (0,T),\\ \rho\,\partial_t \boldsymbol{v} - \mu\,\Delta\boldsymbol{v} + \nabla p &= \rho\,\boldsymbol{f} && \text{in } \Omega_f \times (0,T),\\ \rho\,\partial_t \boldsymbol{v} - \widetilde{\mu}\,\Delta\boldsymbol{v} + \mu\,\boldsymbol{K}^{-1}\cdot\boldsymbol{v} + \nabla p &= \rho\,\phi\,\boldsymbol{f} && \text{in } \Omega_p \times (0,T),\\ (\rho c)^*\,\partial_t\theta + (\rho c)^f\,\boldsymbol{v}\cdot\nabla\theta - \boldsymbol{\nabla}\cdot(\boldsymbol{A}\cdot\nabla\theta) + b\,\theta &= f && \text{in } \Omega_f \cup \Omega_p \times (0,T),\\ \boldsymbol{v}(t=0) &= \boldsymbol{v}_0 && \text{in } \Omega_f \cup \Omega_p,\\ \theta(t=0) &= \theta_0 && \text{in } \Omega_f \cup \Omega_p. \end{aligned}\right. \tag{1}$$

It must be completed by the *first-order algebraic jump boundary conditions (A.J.B.C.) on the interface* Σ obtained by averaging the transport phenomena inside Ω_{fp} from the single-domain model [2, 3], where $\boldsymbol{\sigma}(\boldsymbol{v},p)\cdot\boldsymbol{n} := -p\,\boldsymbol{n} + \widetilde{\mu}\,(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^t)\cdot\boldsymbol{n}$ denotes the stress vector with $\widetilde{\mu} := \mu/\phi$, ϕ being the space variable porosity, and $\boldsymbol{\varphi}(\theta) := -\boldsymbol{A}\cdot\nabla\theta + (\rho c)^f\,\boldsymbol{v}\,\theta$ is the convective-diffusive heat flux vector :

$$\left\{\begin{aligned} [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_\Sigma &= d\,< q > && \text{on } \Sigma \times (0,T),\\ -[\![\boldsymbol{\sigma}(\boldsymbol{v},p)\cdot\boldsymbol{n}]\!]_\Sigma + \mu\,d\,\boldsymbol{K}_\Sigma^{-1}\cdot\overline{\boldsymbol{v}}_\Sigma &= d\,< \rho\,\phi\,\boldsymbol{f} > && \text{on } \Sigma \times (0,T),\\ \overline{\boldsymbol{\sigma}(\boldsymbol{v},p)\cdot\boldsymbol{n}}_\Sigma + \overline{p}_\Sigma\,\boldsymbol{n} - \frac{\mu}{\phi_\Sigma}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\cdot\frac{[\![\boldsymbol{v}]\!]_\Sigma}{d} &= 0 && \text{on } \Sigma \times (0,T),\\ [\![\boldsymbol{\varphi}(\theta)\cdot\boldsymbol{n}]\!]_\Sigma + d\,\left(< b > - (\rho c)^f < q >\right)\overline{\theta}_\Sigma &= d\,< f > && \text{on } \Sigma \times (0,T),\\ \overline{\boldsymbol{\varphi}(\theta)\cdot\boldsymbol{n}}_\Sigma + < A_n >^h\,\frac{[\![\theta]\!]_\Sigma}{d} - (\rho c)^f\,\overline{\boldsymbol{v}\cdot\boldsymbol{n}}_\Sigma\,\overline{\theta}_\Sigma &= 0 && \text{on } \Sigma \times (0,T). \end{aligned}\right. \tag{2}$$

Here we have: $\phi_\Sigma :=< \phi >^h$, $\boldsymbol{K}_\Sigma :=< \boldsymbol{K}(\phi) >^h$, $A_n := \boldsymbol{A}_{nn}$, and $(\rho c)^* := \phi\,(\rho c)^f + (1-\phi)\,(\rho c)^p$.
For a non-centered fictive interface, all the quantities $\overline{\psi}_\Sigma$ are to be replaced by the weighted means :

$$\overline{\psi}^w_\Sigma \;=\; \overline{\psi}_\Sigma + \xi\,[\![\psi]\!]_\Sigma, \qquad \text{with} \qquad -\frac{1}{2} \le \xi := \frac{z_\Sigma}{d} \le \frac{1}{2}. \tag{3}$$

The Stokes/Darcy asymptotic model is obtained by formally letting the effective viscosity $\widetilde{\mu}^p := \widetilde{\mu}_{|\Omega_p} = \varepsilon$ tend to zero. We refer to [8] for the theoretical proof when $\varepsilon \to 0^+$ with the vanishing viscosity method and [10] with the WKB asymptotic expansion to calculate the associated boundary layer.
The equation for the pressure jump $[\![p]\!]_\Sigma$ of the cross-flow through Σ will be detailed during the talk.

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